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A Note on Chaotic Maps

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Abstract. It is well-known that, if a continuous map f of a closed interval into itself has a prime periodic point, then it is chaotic. The converse is not true. In this note, it is shown that a necessary and sufficient condition for f to have a prime periodic point is that it is chaotic and there is a chaotic set consisting of only nonwandering points. Let f be a continuous map of the closed interval I into itself. f is said to be chaotic (in Li-Yorke's sense) if

(1) f has infinitely many periods (of periodic points);

(2) there is an (uncountable) chaotic set $S \subset I$, i.e., an uncountable set $S \subset I$ having the following properties:

(a) for any $x, y \in S$, with $x \neq y$, there exist sequences m_i, n_i, such that $f^{m_i}(x)$ and $f^{m_i}(y)$ converge to the same point but $f^{n_i}(x)$ and $f^{n_i}(y)$ converge to different points,

(b) for every $x \in S$ and every $p \in$ the set $p(f)$ of periodic point of f, there exists a sequence k_i such that $f^{k_i}(x)$ and $f^{k_i}(p)$ converge to different points.

Li and Yorke [4] proved that a sufficient condition for f to be chaotic is that it has a periodic point with period 3. Later, Oono [6] weakened the condition to that f has a prime periodic point, i.e., a periodic point with period not a power of 2. Xiong Jincheng [3] showed by an example that this condition is not necessary. On the other hand, Zhou Zuoling [7] proved a theorem which may be stated in the following form: if f has a prime periodic point, then it is chaotic and the chaotic set S can be taken as a subset of the nonwandering set $\Omega(f)$. The main purpose of the present note is to prove that the converse of this theorem also holds.

We denote the orbit $\{f^n(x) \mid n \geq 0\}$ of an $x \in I$ by $O(x, f)$. y is called an $\omega - limiting$ point of x, if there is a sequence n_i such that $f^{n_i}(x)$ converges to y. The set of all $\omega - limiting$ points of x is denoted by $\omega(x, f)$. x is said to be a chain recurrent point of f, if, for any $\epsilon > 0$, there exist $x_0, x_1, \ldots, x_n$, such that $x_0 = x_n = x$ and $\mid f(x_i) - x_{i+1} \mid < \epsilon$ for $i = 0, 1, \ldots, n-1$. The set of all chain recurrent points of f is denoted by $CR(f)$. f is said to be of type 2^∞, if f has no prime periodic points. $O(x, f)$ is said to be strongly separated under f, if there exists a fixed point z between x and $f(x)$ such that, for all $y \in O(x, f)$, $f(y) > z$ if and only if $y < z$.

Lemma 1. *For any $x \in CR(f) - P(f)$, $O(x, f^n)$ is strongly separated under f $(n = 1, 2, \ldots)$ if and only if f is of type 2^∞.*
For a proof, see reference [2].

Lemma 2. *If f is of type 2^∞, then for any $x \in CR(f) - P(f)$, none of the ω-limiting points of x is periodic.*
For a proof, see reference [2].

Lemma 3. *Let f be of type 2^∞ and let $n > 0$. If $O(x, f^{kn})$ is strongly separated under f^{kn} for $k=1,2$, then the convex hull of $O(x, f^{2n})$ contains no fixed point of f^n.*

For a proof, see reference [5].

Lemma 4. *Let $x_1 \neq x_2$. If for some sequence m_i, $f^{m_i}(x_1)$ and $f^{m_i}(x_2)$ converge to the same point y, then for any $n > 0$, there exists $l > 0$ such that*

$$f^l(y) \in \omega(x_1, f^n) \cap \omega(x_2, f^n).$$

Proof. By hypothesis, y belongs to $\omega(x_1, f)$ and $\omega(x_2, f)$. Since some one of the finitely many residue classes of the integers mod n must contain a sub-sequence of the sequence $m_1, m_2, \ldots$, y must belong to $\omega(f^k(x_1), f^n)$ and $\omega(f^k(x_2), f^n)$ for some $k = 0, 1, \ldots, n-1$. Let $l = n - k$. We have

$$f^l(y) \in f^l[\omega(f^k(x_1), f^n) \cap \omega(f^k(x_2), f^n)] \subset f^l[\omega(f^k(x_1), f^n)] \cap f^l[\omega(f^k(x_2), f^n)],$$

which, by the continuity of f, is contained in

$$\omega(f^{k+l}(x_1), f^n) \cap \omega(f^{k+l}(x_2), f^n) = \omega(f^n(x_1), f^n) \cap \omega(f^n(x_2), f^n) = \omega(x_1, f^n) \cap \omega(x_2, f^n).$$

Lemma 5. *Let f be of type 2^∞, $x \in CR(f) - P(f)$. Then, for any m,n, $m \neq n$, there must be a periodic point between $f^m(x)$ and $f^n(x)$.*

Proof. Suppose $m < n$. Let $q = n - m$, $y = f^m(x)$. Since $x \in CR(f)$, by the continuity of f^m, $y \in CR(f)$. If $y \in P(f)$, then $y \in \omega(x, f)$. By lemma 2, any $\omega - limiting$ point of x cannot be a periodic point, therefore $y \notin P(f)$, and hence $y \in CR(f) - P(f)$. Thus, by lemma 1, there must be a fixed point of f^q between y and $f^q(y)$, which is then a periodic point of f between $f^m(x)$ and $f^n(x)$.

Theorem. *The following three statements are equivalent:*

(1) f has a prime periodic point;

(2) there is a chaodic set $S \subset \Omega(f)$;

(3) there is a chaotic set $S \subset CR(f)$.

Proof. *(1)* implies *(2)*. This is the result in [7] above quoted. *(2)* implies *(3)*. This follows from the fact that $\Omega(f) \subset CR(f)$. *(3)* implies *(1)*. Suppose there is a chaotic set $S \subset CR(f)$. Let $x_1, x_2 \in S$, $x_1 \neq x_2$. By (b) in the definition of chaotic sets, S contains no periodic points. Thus, $x_1, x_2 \in CR(f) - P(f)$. Assume, contrary to *(1)*, f is of type 2^∞.

Case 1. For any $n = 0, 1, 2, \ldots$, there is no periodic points between $f^n(x_1)$ and $f^n(x_2)$. In this case, if $m \neq n$, $f^m(x_1)$ cannot lie between $f^n(x_1)$ and $f^n(x_2)$, for, otherwise, there would be a periodic point between $f^m(x_1)$ and $f^n(x_1)$ by lemma 5, and this periodic point would lie between $f^n(x_1)$ and $f^n(x_2)$. Similarly, if $m \neq n$, $f^m(x_2)$ cannot lie between $f^n(x_1)$ and $f^n(x_2)$ also. It follows that the intervals with end-points $f^n(x_1)$ and $f^n(x_2)$ have no common point for different n's. So, $\mid f^n(x_1) - f^n(x_2) \mid$ must converge to zero, and we cannot find a sequence n_i such that $f^{n_i}(x_1)$ and $f^{n_i}(x_2)$ converge to different points, violating (a) in the definition of chaotic sets.

Case 2. For a certain n_0, there is a periodic point p between $f^{n_0}(x_1)$ and $f^{n_0}(x_2)$. Let t be the period of p. Let $g = f^{n_0}$, $y_1 = g(x_1)$, $y_2 = g(x_2)$. Since f is of type 2^∞, g is also of type 2^∞. Since $x_1, x_2 \in CR(f)$, it follows from the continuity of g, then $y_1, y_2 \in CR(f)$. But $CR(f) \subset CR(g)$, $P(f) = P(g)$. So, $y_1, y_2 \in CR(g) - P(g)$. Hence, by lemma 1, $O(y_1, g^{tk})$ and $O(y_2, g^{tk})$ are strongly separated under g^{tk} for $k = 1, 2$. By lemma 3, the convex hulls of $O(y_1, g^{2t})$ and $O(y_2, g^{2t})$ contain no fixed points of g^t. Since p is a fixed point of g^t, $O(y_1, g^{2t})$ must be on the same side of p, and $O(y_2, g^{2t})$ must be on the other side. Now, by lemma 2, $\omega(y_1, g^{2t})$ and $\omega(y_2, g^{2t})$ contain no periodic point of g, and hence, do not contain p. So,

$$\omega(y_1, g^{2t}) \cap \omega(y_2, g^{2t}) = \emptyset. \qquad (*)$$

Since $x_1, x_2 \in$ the chaotic set S, there is a sequence n_i such that $f^{n_i}(x_1)$ and $f^{n_i}(x_2)$ converge to the same point y. From this, we see that there is a sequence m_i such that $f^{m_i}(y_1)$ and $f^{m_i}(y_2)$ both converge to y. By lemma 4, there is an $l > 0$ such that

$$f^l(y) \in \omega(y_1, f^{2n_0t}) \cap \omega(y_2, f^{2n_0t}),$$

which contradicts $(*)$.

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